International Journal of Engineering, Science and Mathematics
Vol. 9 Issue 4, April 2020,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijesm.co.in, Email: ijesmj@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

FIXED POINT OF STRICTLY PSEUDOCONTRACTIVE MAPPING IN REAL HILBERT SPACE

CHETAN KUMAR SAHU[1], SUBHASH CHANDRA SHRIVASTAVA[2] AND S. BISWAS[3]

*[1*3]Kalinga University Raipur*

[2]Rungta College of Engineering & Technology Bhilai

ABSTRACT: It is known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to study computing the fixed points of strictly pseudocontractive mappings
The purpose of this paper is to study the existence of uniqueness of fixed point for a class of nonlinear mappings defined on real Hilbert space, which among others, contains the class of set of contractive mappings

KEYWORDS: Fixed point, Pseudocontractive mappings, Hilbert space ,Nonlinear integral equation.

1. INTRODUCTION: Most important nonlinear problems of applied mathematics reduce to finding resolutions of nonlinear functional equations (e.g. nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations). It can be formulated in terms of finding the fixed points of a given nonlinear mapping on an infinite dimensional function space X into itself. The theory of fixed point is one of the most powerful tool of modern mathematical analysis. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology & geometry which has many applications in various fields such as mathematics engineering, physics, economics, game theory, biology, chemistry, optimization theory and approximation theory etc. Fixed point theory has its own importance and developed tremendously for the last one and half century.

2. PRELIMINARIES: In the sequel we shall make use of the following notations, definitions, lemmas and theorems.

Definition 2.1 . The mapping T is said to be uniformly L-Lipschitzian, if there exists $L > 0$ such that, for all $x, y \in D(T)$

$$\|T^n x - T^m y\| \leq L\|x - y\|$$

Definition 2.2. T is said to be nonexpansive if for all x, y∈D(T);

The following inequality holds:

$$\|T x - T y\| \leq \|x - y\| \text{ for all } x, y \in D(T);$$

Definition 2.3. T is said to be asymptotically nonexpansive, if there exists a sequence $\{K_n\}$ n≥0 ⊂ [1; ∞) with $\lim_{n\to\infty} K_n = 1$ such that

$$\|T^n x - T^n y\| \leq K_n \|x - y\| \text{ for all } x, y \in D(T);\ n \geq 0:$$

Definition2. 4. Let X be a Banach space, B a closed ball centered at the origin in X, and T: $B \to X$ a pseudo-contractive mapping $(\lambda - 1)\,\|x - y\| \leqslant \|(\lambda I - T)(x) - (\lambda I - T)(y)\|$ for all $x, y \in B$ and $\lambda > 1$.

Definition2. 5. T is said to be asymptotically pseudocontractive, if there exists a sequence $\{K_n\}$ n≥0 ⊂ [1; ∞) with $\lim_{n\to\infty} K_n = 1$ and j(x-y)∈J(x-y) such that

$\langle T^n x - T^n y, j(x - y)\rangle \leq \|x - y\|^2$ for all x, y∈D(T); n ≥ 0

Definition2. 6. A mapping $T : C \to C$ is said to be strictly pseudocontractive if there exists a constant $0 \leq \lambda < 1$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I-T)x-(I-T)y\|$ for all x,y∈C.

Definition2. 7. A mapping T is said to be lie on the ray from the identity mapping I generated by U (denoted by Ray(U)) if there exist constant t > 0 such that T=I+ t (U- I)

Definition2.8. A mapping T:X→X is said to be in class M_2 if there exist a constant β, 0< β<1

Such that $\|Tx-Ty,x-y\| \geq \beta\|Tx - Ty\|^2$ for all x,y∈X.

Remark 1. 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.

2. If T is asymptotically nonexpansive mapping then, for all x, y ∈D(T); there exists j(x - y)∈J(x - y) such that

$\langle T^n x - T^n y, j(x - y)\rangle \leq \|T^n x - T^n y\|\,\|x - y\|$, n ≥ 0:

Hence, every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

Theorem3.1 (a) A mapping U:X→X is strictly pseudocontractive if and only if T=I-U lies in M_2

(b) U is strictly pseudocontractive implies that Ray(U) is strictly pseudocontractive

Proof (a): Suppose that T=I-U∈M_2 therefore

$\|Tx-Ty,x-y\| \geq \beta \|Tx-Ty\|^2$ for all x,y∈X.

Now

$$\|Ux-Uy\|^2 = \|(I-T)x-(I-T)y\|^2$$
$$= \|x-y\|^2 + \|Tx-Ty\|^2 - 2\|Tx-Ty,x-y\|$$
$$\leq \|x-y\|^2 + \|Tx-Ty\|^2 - 2\beta \|Tx-Ty\|^2$$
$$= \|x-y\|^2 + (1-2\beta) \|Tx-Ty\|^2$$
$$= \|x-y\|^2 + k \|Tx-Ty\|^2 \text{ Where } k = (1-2\beta)<1$$
$$= \|x-y\|^2 + k \|(I-U)x-(I-U)y\|^2$$

Which shows that U is strictly pseudocontractive

Conversely, suppose that T=I-U with U is strictly pseudocontractive

$$\text{Therefore } \|Ux-Uy\|^2 \leq \|x-y\|^2 + k \|(I-U)x-(I-U)y\|^2$$
$$= \|x-y\|^2 + k \|Tx-Ty\|^2$$
$$\text{But } \|Ux-Uy\|^2 = \|(I-T)x-(I-T)y\|^2$$
$$\leq \|x-y\|^2 + \|Tx-Ty\|^2 - 2\|Tx-Ty, x-y\|$$
$$\leq \|x-y\|^2 + k\|Tx-Ty\|^2$$

Which implies $\|Tx-Ty, x-y\| \geq (\frac{1-K}{2}) \|Tx-Ty\|^2$

This shows that $T \in M_2$ with $\beta = (\frac{1-K}{2})$

(b) Let U be pseudocontractive, which implies that I-U∈M_2, if the mapping W is lie from identity mapping I generated by U then there exist t>0 such that

$$W = I + t\,(U-I)$$
$$\Rightarrow t\,(I-U) = I-W$$

Now $I-U \in M_2$

$$\Rightarrow t(I-U) \in M_2$$
$$\Rightarrow I-W \in M_2$$

⇒ W is strictly pseudocontractive by part (a)

Theorem 3.2 U is strictly pseudo contractive if and only if there exist an element W∈Ray(U) such that W is nonexpansive.

Proof: Suppose there exist W∈Ray(U) such that W is nonexpansive then there exist t>1 such that W=I+t(U-I) which implies. Now W=I+t(U-I)

$\Rightarrow$I-W=t(I-U)

$\Rightarrow$I-W= tT where T=I-U

Since W is nonexpansive therefore $\|Wx-Wy\| \leq \|x-y\|$

$$\Rightarrow \|Wx - Wy\|^2 \leq \|x - y\|^2$$

$$\Rightarrow \|(I - tT)x - (I - tT)y\|^2 \leq \|x - y\|^2$$

$$\Rightarrow \|x - y\|^2 + \|tTx - tTy\|^2 - 2\|tTx\text{-}tTy,x\text{-}y\| \leq \|x - y\|^2$$

$$\Rightarrow \|tTx - tTy\|^2 - 2\|tTx\text{-}tTy,x\text{-}y\| \leq 0$$

$$\Rightarrow \|Tx\text{-}Ty,x\text{-}y\| \geq \beta\|Tx - Ty\|^2 \text{for some } 0< \beta<1$$

$$\Rightarrow T=I-U \in M_2$$

$\Rightarrow$ U is strictly pseudocontractive by theorem 3.1 (a)

Conversely, suppose U is strictly pseudocontractive. By theorem 3.1 T=I-U$\in M_2$and hence

$$\|Tx\text{-}Ty,\ x\text{-}y\| \geq \beta\|Tx - Ty\|^2 \text{for for some } \beta>0$$

Consider the mapping U_t= I+ t(U-I) in Ray(U) for t>0

Now U_t= I+ t(U-I)

$\Rightarrow$ U_t= I- tT it follows that

$$\|U_tx - U_ty\|^2 = \|(I - tT)x - (I - tT)y\|^2$$

$$= \|x - y\|^2 + t^2\|Tx - Ty\|^2 \text{-}2t\|Tx\text{-}Ty,x\text{-}y\|$$

$$\leq \|x - y\|^2 + t^2\|Tx - Ty\|^2 \text{-}2t\beta\|Tx - Ty\|^2$$

$$= \|x - y\|^2 + (t^2 \text{-}2t\beta)\|Tx - Ty\|^2$$

$$\leq \|x - y\|^2 \text{for any fixed t with } 0 < t \leq 2\beta=1-k$$

Thus $\|Wx - Wy\|^2 \leq \|x - y\|^2$ for W= U_t

$$\Rightarrow \|Wx\text{-}Wy\| \leq \|x\text{-}y\|$$

$\Rightarrow$ W is nonexpansive

OUR MAIN RESULT IS AS FOLLOWS:

Theorem.Let C be a bounded, closed, convex weakly sequently compact subset of X and U a strictly pseudocontractive mapping of C into C i.e. there exist a constant k<1

$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I-U)x-(I-U)y\|$ for all x, y∈C. Then for each $x_0 \in C$ and any fixed γ such that 1-k < γ < 1, $U_\gamma^n x_0 \rightarrow y \in C$(weakly) and y is a fixed point of U in C.If additionally we assume that U is demicompact then $U_\gamma^n x_0 \rightarrow y$ (strongly)

Proof: Since U is strictly pseudocontractive therefore by theorem 3.2 shows that for every fixed t such that 0< t≤ k-1the mapping U_t= I+ t(U-I) = tU+ (I-t)I is nonexpansive. By Hicks and Huffmann second theorem we know that if any mapping f is nonexpansive in bounded, closed, convex sequently compact set then it has a fixed point therefore U_t has a fixed point in C. Let it be z∈C then $U_t(z)$=z. This implies that

$$[\ tU+ (I-t)I]z = z$$

$$\Rightarrow tU(z) + (I-t)I(z) = z$$

$$\Rightarrow tU(z) +(I-t)z = z$$

$$\Rightarrow tU(z) +z- tz = z$$

$$\Rightarrow tU(z) = tz$$

$$\Rightarrow U(z) = z$$

Which shows that z is a fixed point of U in C

Further by Hicks and Huffmann ninth theorem for any $x_0 \in C$ and any fixed λ with 0 < λ <1 the sequence $\{(U_t)^n x_0\}$ converges weakly to some point y on U in C. But

$$(U_t)_\lambda = \lambda I + (1-\lambda) U_t$$

$$= \lambda I + (1-\lambda) [I+ t(U-I)]$$

$$= (1-\lambda)tU + 1 - (1-\lambda)t\, I$$

$$= \gamma I + (I-\gamma)U \text{ where } \gamma = 1-(1-\lambda)t$$

$$= U_\gamma$$

Now γ = 1- (1- λ)t implies that γ < 1 also t ≤ k-1 1 – λ < 1

$$\Rightarrow t(1-\lambda) < k-1$$

$$\Rightarrow 1-k < -t(1-\lambda)$$

$$\Rightarrow 1-k < 1- t(1-\lambda)$$

$$\Rightarrow 1-k < \gamma$$

$$\Rightarrow 1-k < \gamma < 1$$

This completes the proof of first part of the theorem

To prove the second part we will use the the result of seventh theorem of Hicks and Huffmann theorem and we need only to show that U is demi-compact. Now

$$U_\gamma x - x = [\gamma I + (I-\gamma)U]x - x$$

$$= \gamma x - x + (I-\gamma)Ux$$

$$=(I - \gamma)(Ux - x) \text{ for all } x \in C$$

The demi-compectness of U and above equality proves the second part of the theorem

CONCLUSION:

Finding fixed points of nonlinear mappings especially, nonexpansive mappings has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is well known that strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems. In this paper, we devote to construct the

methods for computing the fixed points of strictly pseudocontractive mappings.

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